restriction of the extent of blowing, and the step-wise character of the blowing-suction distribution which necessitates less blowing and more suction than the optimum distribution, reduce the momentum-thickness Reynolds number and bring the boundary-layer velocity profiles closer to the asymptotic suction profile ($K = 0$). The change in profile results in a larger shape factor for the measured velocity profiles.

The results of Figs. 5–10 correspond to transition about 5 in. upstream of the end of the model. Any attempt to establish transition at the very end of the model would be confused by disturbances of the wake of the model. For convenience in detecting the nature of the boundary-layer flow, the transition point, rather than fully laminar flow, was established at the chosen location. Judgment of a fully laminar flow, just devoid of turbulence, involves the risk of establishing a subcritical laminar boundary layer.

The combination of suction applied to maintain a laminar boundary layer and blowing at the stagnation point to improve the tolerance of the boundary layer to roughness should be ideal for substantially reducing skin friction of bodies at high Reynolds numbers.

With an appropriate choice of the blowing rate, the fluid removed by suction may be the same fluid injected near the stagnation point. For example, for a body with a 4:1 elliptical-nose and a length-to-thickness ratio of 15, the amounts of the injected and the sucked fluid are equal for a blowing rate of 1.2% at the stagnation point. This procedure would reduce the danger of clogging the porous skin in flights through turbid fluids, since most of the sucked fluid is composed of the fluid ejected at the nose, and eliminate the necessity of disposing of the sucked fluid.

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Similar Solutions of the Boundary-Layer Equations for a Non-Newtonian Fluid

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Nomenclature

c_p = constant pressure specific heat
k = thermal conductivity
K,N = constants in the power-law shear model
r = constant in Ref. 7 describing the required temperature distribution for similar solutions
T = temperature
u,v = velocity in the x and y direction, respectively
ρ = density

Subscripts

L = characteristic or reference length
∞ = reference (freestream) conditions
w = wall conditions

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Introduction

THE mathematical simplifications to the boundary-layer equations which are possible when the concept of similar solutions is utilized are well known from classical Newtonian flow[1,2] and will not be reiterated in this note. Schowalter[3] and Acrivos, Shah, and Petersen[4] were among the first to publish the results of investigations of the flow of a non-Newtonian fluid past an external surface. Schowalter's primary objective was to determine those flows of a power-law non-Newtonian fluid for which similar solutions could be obtained for the momentum equation. Acrivos et al. obtained numerical solutions of the momentum equation for the flow of the same type of fluid past a horizontal flat plate. The heat-transfer coefficient was also obtained by considering an asymptotic form of the energy equation valid in the limiting case of large Prandtl numbers.

Among the investigators that have considered the subject of similar solutions of the boundary-layer equations for external flow of non-Newtonian fluids since the investigations of Schowalter and Acrivos, Shah, and Petersen are the authors of Refs. 5–7. Only in the papers by Acrivos et al.[4] and Lee and Ames[7] was the energy equation considered in the investigation to determine the possibility of obtaining similar solutions. The authors of the former paper concluded that similar solutions did not exist for the energy equation for the case of the flow of a power-law non-Newtonian fluid past a flat plate with forced convection and that finite-difference techniques must be used to obtain a solution. A similar conclusion was reported in the latter paper, however, similar solutions for the energy equation were found to be possible when a temperature dependent thermal conductivity was postulated. The solutions obtained in this manner imposed a restriction on the surface temperature distribution of the form $\theta_w = (x)^{(N-1)/[(r-1)(N+1)]}$. Although similar solutions were obtained for this special case, the correctness of an assumption of a constant viscosity but a temperature-dependent thermal conductivity seems questionable.

The purpose of this Note is to present a formulation of the boundary-layer equations which yields similar solutions for both the momentum and energy equation for the flow of non-Newtonian fluid past a horizontal flat plate. The momentum equation is first transformed using the usual definition of the similarity variables. The energy equation is then transformed using a slightly different definition of the independent similarity variable. The transformed momentum equation is completely similar; however, the transformed energy is only locally similar, since the axial distance along the plate must be specified to obtain numerical solutions.

The flow is assumed to be laminar and all fluid properties are constant. The fluid is assumed to be characterized by the well-known power-law relationship between the stress and rate of deformation tensors.

Mathematical Formulation

The two-dimensional boundary-layer equations for a constant property power-law, non-Newtonian flow, without viscous dissipation, are

Continuity:

$$\partial u/\partial x + \partial v/\partial y = 0 \tag{1}$$

x-momentum:

$$\rho u(\partial u/\partial x) + \rho v(\partial u/\partial y) = (\partial/\partial y)(\tau_{xy}) \tag{2}$$

Energy:

$$\rho u c_p(\partial T/\partial x) + \rho v c_p(\partial T/\partial y) = (\partial/\partial y)(k\,\partial T/\partial y) \tag{3}$$

Shear model:

$$\tau_{xy} = K(\partial u/\partial y)^N \tag{4}$$

The nondimensional quantities

$$\bar{x} = x/L; \quad \bar{y} = y/L(Re_L)^{1/1+N}; \quad Re_L = [\rho U_\infty^{(2-N)}L^N]/K$$

$$\bar{u} = u/U_\infty; \quad \bar{v} = v/U_\infty(Re_L)^{1/1+N}$$

$$\theta = (T - T_w)/(T_\infty - T_w)$$

and the assumption of constant properties are introduced in Eqs. (1–4) to obtain

$$\partial\bar{u}/\partial\bar{x} + \partial\bar{v}/\partial\bar{y} = 0 \tag{5}$$

$$\bar{u}(\partial\bar{u}/\partial\bar{x}) + \bar{v}(\partial\bar{u}/\partial\bar{y}) = (\partial/\partial\bar{y})(\partial\bar{u}/\partial\bar{y})^N \tag{6}$$

$$\bar{u}(\partial\theta/\partial\bar{x}) + \bar{v}(\partial\theta/\partial\bar{y}) = (1/Pr)(\partial^2\theta/\partial\bar{y}^2) \tag{7}$$

where the Prandtl number is defined as

$$\mathrm{Pr} = (c_p U_\infty \rho L/k)(1/Re_L)^{2/1+N}$$

Defining the usual similarity variables $\eta = \bar{y}/(2x)^{1/1+N}$ and $f'(\eta) = u/U_\infty$, where primes denote differentiation with respect to the independent variable, and combining Eqs. (5) and (6) yields the Blasius-type nonlinear ordinary differential equation

$$f'''(\eta) + [2/N(1+N)]f(\eta)[f''(\eta)]^{2-N} = 0 \tag{8}$$

with boundary conditions

$$f(\eta) = f_w,\ f'(\eta) = 0 \qquad @\ \eta = 0$$

$$\lim f'(\eta) = 1 \qquad @\ \eta \to \infty$$

where mass injection at the surface corresponds to $f_w < 0$ and suction to $f_w > 0$.

It is easily shown that Eq. (7) will not be properly transformed using η as the similarity variable. In the past, this has been the termination point in attempts to obtain similarity solutions for the energy equation, however, this limitation can be relaxed using a slightly different definition of the independent similarity variable.

The similarity variables $f'(\eta)$ and η are replaced by $f'(\xi) = u/U_\infty$ and $\xi = \bar{y}/(2\bar{x})^{1/2}$, respectively, and $\bar{v}$ is obtained by integrating Eq. (5);

$$\bar{v} = [1/(2\bar{x})^{1/2}][\xi f'(\xi) - f(\xi)] \tag{9}$$

Now Eq. (7) can be written in terms of the new similarity variables to obtain

$$\theta''(\xi) + \mathrm{Pr}f(\xi)\theta'(\xi) = 0 \tag{10}$$

with boundary conditions

$$\theta(\xi) = 0 \qquad @\ \xi = 0$$

$$\lim \theta(\xi) = 1 \qquad @\ \xi \to \infty$$

Equations (8) and (10), together with the boundary conditions, represent the transformed mathematical system. It is to be noted that η and ξ are explicitly related by

$$\eta = (2x/L)^{(1-N)/2(1+N)}\xi \tag{11}$$

Thus, with $f(\eta)$ for all η available from the solution of Eq. (8), it is an easy task to obtain the corresponding values of $f(\xi)$, for any specified axial distance, which are required to solve Eq. (10). Thus, the straightforward techniques of solving the classical incompressible Newtonian fluid boundary-layer equations are now seen to be applicable to the present system of equations.

Summary

The two-dimensional boundary-layer equations for the flow of a laminar, power-law non-Newtonian fluid with constant properties can be transformed, using appropriately defined similarity variables, into a form suitable for obtaining similar solutions. The transformed momentum equation is completely similar; however, the x dependence in the $\eta - \xi$ relation limits the energy equation to local similarity.

The requirement for the postulate made by Lee and Ames[7] of a constant viscosity and a temperature-dependent thermal conductivity in order to use the concept of similarity to obtain solutions of the energy equation is thus relaxed and the usual condition of an isothermal wall is easily solved for all values of N. That the transformed equations have the proper form for the special case of a Newtonian fluid can be shown by letting $N = 1$ in Eqs. (8) and (11).

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